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MIXED FINITE ELEMENT METHODS
FOR ELLIPTIC EQUATIONS

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ABSTRACT

Necessary and sufficient conditions are given for the convergence of mixed methods for the solution of elliptic differential equations. Error bounds are given in the L^2 norm which show that full accuracy can be obtained when all variables are approximated by piecewise polynomial functions of the same degree.

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1. Introduction

It is easy to show that the functional

$$J(u,\sigma) = \tfrac{1}{2}(\sigma,\sigma) + (Tu-\sigma,\sigma) - (f,u) \tag{1}$$

assumes a stationary value whenever (u,σ) satisfies the equations

$$\begin{aligned} &Tu = \sigma \quad \text{in } \Omega , \qquad bu = 0 \quad \text{on } \partial\Omega \\ &T^*\sigma = f \end{aligned} \tag{2}$$

where T is a linear map from the Hilbert space H_1 to the Hilbert space H_2, T^* is its formal adjoint, and b is a boundary operator. Mixed methods result from the application of Galerkin techniques to the solution of (2), or equivalently to the first variation of (1). The purpose of this note is to give necessary and sufficient conditions for the convergence of mixed methods for the solution of elliptic equations. These results extend and sharpen earlier results [7], [8], [9]. With mixed methods several dependent variables are approximated simultaneously. We also give error bounds which show that full accuracy can be obtained for all variables being approximated. To make exposition easier we discuss our results using the model second order elliptic equation

$$\begin{aligned} \nabla u &= \underline{\sigma} \quad \text{in } \Omega , \\ -\text{div}\, \underline{\sigma} &= f \quad \text{in } \Omega , \\ u &= 0 \quad \text{on } \partial\Omega . \end{aligned} \tag{3}$$

Our results can be immediately extended to general second order elliptic equations and to the elliptic systems which arise in linear elasticity. Our results can also be extended to higher order equations. The application to fourth order equations does not directly eliminate the need to seek finite element approximations in C^1. In order to be able to seek approximations in C^0, it seems necessary to use hybrid methods or to use a technique like that used by Ciarlet and Raviart [5] which introduces an additional variable into the calculations.

2. Main Results

We introduce the space

$$H(\mathrm{div};\Omega) = \{\underline{v} \in (L^2(\Omega))^2 \; ; \; \mathrm{div}\, \underline{v} \in L^2(\Omega)\} \tag{4}$$

provided with the norm

$$\|\underline{v}\|_{H(\mathrm{div};\Omega)} = (\|\underline{v}\|^2_{L^2(\Omega)} + \|\mathrm{div}\, \underline{v}\|^2_{L^2(\Omega)})^{\frac{1}{2}} \, .$$

Note that $(H^1(\Omega))^2 \subset H(\mathrm{div};\Omega)$. Let $W = H^1_0(\Omega) \times H(\mathrm{div};\Omega)$ and define on W the norm

$$\|\underline{w}\|_W = (\|u\|^2_{H^1(\Omega)} + \|\underline{\sigma}\|^2_{H(\mathrm{div};\Omega)})^{\frac{1}{2}} \tag{5}$$

where $\underline{w} = (u,\underline{\sigma})$.

Choose finite dimensional spaces $V_h \subset H^1_0(\Omega)$ and $S_h \subset H(\mathrm{div};\Omega)$, and define $W_h = V_h \times S_h$. The mixed method we consider for the approximate solution of (3) is: find $\underline{w}_h = (u_h,\underline{\sigma}_h) \in W_h$ such that

$$B(\underline{w}_h,\bar{\underline{w}}_h) = \int_\Omega f\,\bar{u}_h \, , \qquad \text{all } \bar{\underline{w}}_h \in W_h \, , \tag{6}$$

where

$$B(\underline{w}_h,\bar{\underline{w}}_h) = \int_\Omega \nabla u_h\, \bar{\underline{\sigma}}_h + \underline{\sigma}_h \nabla \bar{u}_h - \underline{\sigma}_h \bar{\underline{\sigma}}_h \, . \tag{7}$$

We make the following standard approximability assumptions

(8) There is some $r_1 \geq 2$ such that if $u \in H^s(\Omega)$, $2 \leq s \leq r_1$, then there is some $v_h \in V_h$ such that

$$||u-v_h||_{H^k(\Omega)} \leq C\, h^{s-k}||u||_{H^s(\Omega)} \quad , \qquad k = 0,1,$$

where C is independent of u and v_h.

(9) There is some $r_2 \geq 2$ such that if $\underset{\sim}{\sigma} \in (H^t(\Omega))^2$, $2 \leq t \leq r_2$, then there is some $\underset{\sim}{\rho}_h \in S_h$ such that

$$||\underset{\sim}{\sigma} - \underset{\sim}{\rho}_h||_{L^2(\Omega)} \leq C\, h^t||\underset{\sim}{\sigma}||_{(H^t(\Omega))^2} ,$$

$$||\operatorname{div}(\underset{\sim}{\sigma} - \underset{\sim}{\rho}_h)||_{L^2(\Omega)} \leq C\, h^{t-1}||\underset{\sim}{\sigma}||_{(H^t(\Omega))^2} ,$$

where C is independent of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\rho}_h$.

Let π_h denote the orthogonal projection of $L^2(\Omega)$ onto V_h and P_h denote the orthogonal projection of $(L^2(\Omega))^2$ onto S_h.

<u>Theorem 1.</u> Suppose that the finite dimensional spaces $V_h \subset H_0^1(\Omega)$ and $S_h \subset H(\operatorname{div};\Omega)$ satisfy (8), (9) along with

(10) $$\lim_{h \to 0} ||\nabla \bar{u}_h - P_h \nabla \bar{u}_h||_{L^2(\Omega)} \to 0 \quad , \qquad \text{all } \bar{u}_h \in V_h ,$$

(11) $$\lim_{h \to 0} ||\operatorname{div} \bar{\underset{\sim}{\sigma}}_h - \pi_h \operatorname{div} \bar{\underset{\sim}{\sigma}}_h||_{(L^2(\Omega))^2} \to 0, \qquad \text{all } \bar{\underset{\sim}{\sigma}}_h \in S_h .$$

Let $(u,\underline{\sigma})$ be the unique solution to (3) where $f \varepsilon H^k(\Omega)$, $k \geq 0$. Then for h sufficiently small, (6) has the unique solution $(u_h,\underline{\sigma}_h)$ and

$$(12) \qquad \left(||u-u_h||^2_{H^1(\Omega)} + ||\underline{\sigma}-\underline{\sigma}_h||^2_{H(\mathrm{div};\Omega)} \right)^{1/2}$$

$$\leq C\left(h^{s-1}||u||_{H^s(\Omega)} + h^{t-1}||\underline{\sigma}||_{(H^t(\Omega))^2} \right).$$

Proof. We show that if the spaces V_h and S_h satisfy (10) and (11), then for h sufficiently small, there exists a constant $\alpha_0 > 0$ such that for all $\underline{w}_h \varepsilon W_h$

$$(13) \qquad \sup_{\underline{\bar{w}}_h \varepsilon W_h} B(\underline{w}_h,\underline{\bar{w}}_h) \geq \alpha||\underline{w}_h||_W \, ||\underline{\bar{w}}_h||_W \, .$$

By Babuska [1, Theorem 2.2] (see also [2, Theorem 6.2.1, page 186]), the fact that $B(\underline{w},\underline{\bar{w}})$ is symmetric and bounded, (i.e., there exists a constant M independent of w and $\bar{w}$ such that

$$B(w,\bar{w}) \leq M||w||_W||\bar{w}||_W \quad , \qquad \text{all } w,\bar{w} \,\varepsilon\, W,) \quad ,$$

along with (13) are sufficient to insure that (6) has a unique solution.

Given $\underline{w}_h = (u_h,\underline{\sigma}_h)$, we choose

$$\bar{u}_h = 2u_h - \pi_h(\mathrm{div}\underline{\sigma}_h) \qquad , \qquad \underline{\bar{\sigma}}_h = -\underline{\sigma}_h + P_h(\nabla u_h) \ .$$

Clearly $\underline{\bar{w}}_h = (\bar{u}_h,\underline{\bar{\sigma}}_h) \,\varepsilon\, W_h$ and there exists a constant C such that

$$||\underline{\bar{w}}_h||_W \leq C \, ||\underline{w}_h||_W \, .$$

We obtain

$$\sup_{\hat{\underline{w}}_h \in \underline{W}_h} B(\underline{w}_h,\hat{\underline{w}}_h) \geq B(\underline{w}_h,\bar{\underline{w}}_h) = ||\underline{\sigma}_h||^2_{L^2(\Omega)} + ||\mathrm{div}\,\underline{\sigma}_h||^2_{L^2(\Omega)}$$

$$(14) \qquad + ||\nabla u_h||^2_{L^2(\Omega)} - \int_\Omega \mathrm{div}\,\underline{\sigma}_h\big(\mathrm{div}\,\underline{\sigma}_h - \pi_h(\mathrm{div}\,\underline{\sigma}_h)\big)$$

$$- \int_\Omega \nabla u_h\big(\nabla u_h - P_h(\nabla u_h)\big) \ .$$

Since for functions in $H^1_0(\Omega)$, $||\nabla v||_{L^2(\Omega)}$ is equivalent to the usual H^1 norm, we obtain

$$B(w_h,\bar{w}_h) \geq \frac{1}{C}||\bar{\underline{w}}_h||_W||\underline{w}_h||_W\Big(1-||\mathrm{div}\,\underline{\sigma}_h-\pi_h(\mathrm{div}\,\underline{\sigma}_h)||_{L^2(\Omega)} - ||\nabla u_h - P_h(\nabla u_h)||_{L^2(\Omega)}\Big)$$

$$\geq \alpha_0||\underline{w}_h||_W||\bar{\underline{w}}_h||_W$$

for h sufficiently small.

We may again use Babuska's theorem [1, Theorem 2.2] to obtain the error bound

$$(15) \qquad ||u-u_h||_{H^1(\Omega)} + ||\underline{\sigma}-\underline{\sigma}_h||_{H(\mathrm{div};\Omega)}$$

$$\leq \inf_{v_h \varepsilon V_h} ||u-v_h||_{H^1(\Omega)} + \inf_{\underline{\rho}_h \varepsilon S_h} ||\underline{\sigma}-\underline{\rho}_h||_{H(\mathrm{div};\Omega)} \ .$$

The bound (12) follows immediately from (15) by (8) and (9).

The problem (6) is equivalent to a system of linear algebraic equations. Let $\phi_1,\ldots,\phi_n$ be a basis for V_h and $\underline{\psi}_1,\ldots,\underline{\psi}_m$ be a basis for S_h. Then

$$u_h(\underline{x}) = \sum_{i=1}^n u_i\phi_i(\underline{x}) \quad , \quad \underline{\sigma}_h(\underline{x}) = \sum_{i=1}^m \sigma_i\underline{\psi}_i(\underline{x}) \quad ,$$

and the vector of weights $\underline{u} = (u_j)$, $\underline{\sigma} = (\sigma_j)$ are computed from

$$\begin{bmatrix} A & M \\ M^T & 0 \end{bmatrix} \begin{bmatrix} \underline{\sigma} \\ \underline{u} \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{f} \end{bmatrix} \tag{16}$$

where A is negative definite. Thus the system (16) is nonsingular if and only if the $m \times n$ matrix M with entries

$$\int_\Omega \underline{\psi}_i \nabla\phi_j \qquad \begin{matrix} i = 1,\ldots,m, \\ j = 1,\ldots,n, \end{matrix}$$

is of full rank. This is a weaker condition than (10) and (11). It is easy to see, however, that (10) is necessary in order to get convergent approximations. Suppose the $\bar{u}_h$ in (10) is the finite element approximation from V_h to the solution u of

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega , \\ u &= 0 \qquad \text{on } \Omega , \end{aligned}$$

computed using the standard finite element method. The standard error analysis shows that $\nabla\bar{u}_h$ converges to $\nabla u = \underline{\sigma}$ as $h \to 0$. Thus (10) is a necessary condition for the mixed finite element approximation $\underline{\sigma}_h$ to converge to $\underline{\sigma}$. A necessary condition for the satisfaction of (10) is that $\dim S_h \geq \dim \nabla(V_h)$, where $\nabla(V_h)$ is the space of all gradients of elements of V_h.

The condition (11) was used in our proof of Theorem 1 to introduce the term $||\text{div}\, \underline{\sigma}_h||_{L^2(\Omega)}$ into the right side of (14) as required in our norm (5) for W. If we had used instead the weaker norm

$$|||\underline{w}|||_W = (||u||^2_{H^1(\Omega)} + ||\underline{\sigma}||^2_{L^2(\Omega)})^{\frac{1}{2}} , \tag{17}$$

we would not have needed (11). We have included the term $||\text{div}\,\sigma||_{L^2(\Omega)}$ in our norm on W so that (in Theorem 2) we can use Nitsche's trick to prove full accuracy in the L^2 norm for both the approximations u_h and $\underset{\sim}{\sigma}_h$ when the spaces V_h and S_h are the piecewise polynomials of the same degree.

Suppose Ω has a smooth boundary, $f \in H^k(\Omega)$, $k \geq 1$, and let V_h^+ be the same subspace as V_h except that elements of V_h^+ are not required to satisfy the boundary condition $u = 0$ on $\partial\Omega$. For subspaces V_h^+ used in practice one can smooth $f = -\text{div}\,\underset{\sim}{\sigma}$ (if necessary) to obtain $\tilde{f}$ and then define an interpolant $\tilde{f}_I$ to $\tilde{f}$ such that

$$||f-\tilde{f}_I||_{L^2(\Omega)} \leq C h ||f||_{H^1(\Omega)} \;.$$

Let $\tilde{f}^0$ be the same as $\tilde{f}_I$ but with boundary nodes set to zero. Then $\tilde{f}_I^0 \in V_h$ and it can be shown that

$$||\tilde{f}_I-\tilde{f}_I^0||_{L^2(\Omega)} \leq C h^{\frac{1}{2}} ||f||_{H^1(\Omega)}$$

(A similar calculation is used to relate the change in solution due to a change in domain in [10, pages 194-195].) Here we have used the fact [11,page 237] that $\tilde{f}$ can be chosen to satisfy

$$||\tilde{f}||_{H^2(\Omega)} \leq C h^{-1} ||f||_{H^1(\Omega)} \;.$$

Thus

$$||\text{div}\,\underset{\sim}{\sigma} - \pi_h \text{div}\,\underset{\sim}{\sigma}||_{L^2(\Omega)} \to 0$$

as $h \to 0$ so that (11) is necessary for div $\underset{\sim}{\sigma}_h$ to converge to div $\underset{\sim}{\sigma}$ in L^2.

Other authors have also given conditions for the convergence of mixed finite element methods. Oden and Lee [7] have essentially the same condition as (10) for approximations in linear elasticity using a norm on W equivalent to (17). They use Nitsche's trick with the assumption that $\nabla \bar{u}_h \varepsilon S_h$ to obtain optimal bounds in L^2 for the finite element approximation u_h. Except for a remark similar to ours concerning the necessary conditions for the system (16) to be nonsingular, they make no further observations concerning the necessity of their condition. Oden and Reddy in [8] give optimal bounds in L^2 for both u_h and $\underline{\sigma}_h$ but give conditions on the subspaces V_h and S_h which are very difficult to check.

Raviart and Thomas [9] also consider mixed methods for second order elliptic equations. They seek approximations in the space $\tilde{W} = L^2(\Omega) \times H(\text{div};\Omega)$ with norm

$$||\underline{w}||_{\tilde{W}} = (||u||^2_{L^2(\Omega)} + ||\underline{\sigma}||^2_{H(\text{div};\Omega)})^{\frac{1}{2}}$$

where $\underline{w} = (u,\underline{\sigma})$. They give sufficient conditions for existence and uniqueness derived from Brezzi's results [3]. The finite elements they construct, which satisfy their conditions, also trivially satisfy (11). The condition (10) is not needed for their norm.

We now derive estimates in L^2 for both u_h and $\underline{\sigma}_h$. Let $e_u = u - u_h$, $e_{\underline{\sigma}} = \underline{\sigma}-\underline{\sigma}_h$, and $\underline{e} = (e_u,e_{\underline{\sigma}})$.

<u>Theorem 2.</u> Assume the region Ω is convex. Then

(18) $$||e_u||_{L^2(\Omega)} \leq Ch||\underline{e}||_W$$

(19) $$||e_{\underline{\sigma}}||_{L^2(\Omega)} \leq Ch||\underline{e}||_W \, .$$

Proof. The proof uses the idea of Nitsche's trick [6]. Let $(\phi,\underline{v})$ be the solution to

$$\nabla\phi = \underline{v} \quad \text{in } \Omega$$
$$-\operatorname{div} \underline{v} = e_u \quad \text{in } \Omega$$
$$\phi = 0 \quad \text{on } \partial\Omega .$$

Then

$$(e_u,e_u) = (e_u,\operatorname{div} \underline{v}) = (-\nabla e_u,\underline{v}-\underline{v}_h) + (e_{\underline{\sigma}},\underline{v}_h-\nabla\bar{u}_h) ,$$

for all $\underline{v}_h \in S_h$, $\bar{u}_h \in V_h$. Choose $\underline{v}_h \in S_h$ such that

$$||\underline{v}-\underline{v}_h||_{L^2(\Omega)} \leq Ch||\nabla\phi||_{H^1(\Omega)} \leq C'h||\phi||_{H^2(\Omega)}$$

and $\bar{u}_h$ such that

$$||\nabla\bar{u}_h-\nabla\phi||_{L^2(\Omega)} \leq Ch||\phi||_{H^2(\Omega)} .$$

Using

$$||\phi||_{H^2(\Omega)} \leq K||e_u||_{L^2(\Omega)}$$

and the Schwartz inequality we obtain (18).

Now let ψ be the solution of

$$-\Delta\psi = \operatorname{div} e_{\underline{\sigma}} \quad \text{in } \Omega$$
$$\psi = 0 \quad \text{on } \partial\Omega$$

so that $\nabla\psi = e_{\underline{\sigma}}$. Then

$$(\underset{=}{e}_{\sigma},\underset{=}{e}_{\sigma}) = (\underset{=}{e}_{\sigma},\nabla\psi - \nabla\psi_h) \quad , \quad \text{all} \quad \psi_h \in V_h \ .$$

Choose $\psi_h \in V_h$ such that

$$||\nabla\psi-\nabla\psi_h|| \leq Ch||\psi||_{H^2(\Omega)} \leq C'h||\text{div } \underset{=}{e}_{\sigma}||_{L^2(\Omega)} \quad ,$$

from which (19) follows.

If the boundary of Ω is curved, isoparametric elements can be used on boundary triangles. It is straightforward to check that the orders of accuracy presented here are not reduced when isoparametric elements are used in boundary triangles similarly to the results of Ciarlet and Raviart [4] and Zlámal [11], [12] for the standard finite element approximations to second order elliptic problems. We note that since no more than one partial derivative appears in any integrand, numerical integration is not necessary to integrate the isoparametric elements.

References

[1] Babuska, I., "Error bounds for finite element method," Num. Math., 16 (1971), 322-333.

[2] Babuska, I. and Aziz, A. K., "Survey lectures on the mathematical theory of finite elements," in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, ed., Academic Press, New York, 1972, 3-395.

[3] Brezzi, F., "On the existence, uniqueness, and approximation of saddle-point problems arising from Lagrangian multipliers," R.A.I.R.O. 8 (1974), 129-151.

[4] Ciarlet, P. G. and Raviart, P. A., "The combined effect of curved boundaries and numerical integration in isoparametric finite element methods," in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, ed., Academic Press, New York, 1972, 409-474.

[5] Ciarlet, P. G. and Raviart, P. A., "A mixed finite element method for the biharmonic equation," in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., Academic Press, New York, 1974, 125-145.

[6] Nitsche, J., "Ein Kriterium fur die Quasi-Optimalitat des Ritzchen Verfahrens," Num. Math., 11 (1968), 346-48.

[7] Oden, J. T. and Lee, J. K., "Theory of mixed and hybrid finite element approximations in linear elasticity," Symposium on Application of Methods of Functional Analysis to Problems of Mechanics, Marseille, France, 1975.

[8] Oden, J. T. and Reddy, J. N., "On mixed finite element approximations," SIAM J. Num. Anal.,13 (1976), 393-404.

[9] Raviart, P. A. and Thomas, J. M., "A mixed finite element method for second order elliptic problems," (to appear).

[10] Strang, G. and Fix, G. J., An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[11] Zlámal, M., "Curved elements in the finite element method I," SIAM J. Num. Anal., 10 (1973), 229-240.

[12] Zlámal, M., "Curved elements in the finite element method II," SIAM J. Num. Anal., 11 (1974), 347-362.